CONDUCTORS OF WILD EXTENSIONS OF LOCAL FIELDS, ESPECIALLY IN MIXED CHARACTERISTIC $(0,2)$

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ABSTRACT. If K_0 is the fraction field of the Witt vectors over an algebraically closed field k of characteristic p, we calculate upper bounds on the conductor of higher ramification for (the Galois closure of) extensions of the form $K_0(\zeta_{p^r}, \sqrt[p^r]{a})/K_0$, where $a \in K_0(\zeta_{p^r})$. Here ζ_{p^r} is a primitive p^rth root of unity. In certain cases, including when $a \in K_0$ and $p = 2$, we calculate the conductor exactly. These calculations can be used to determine the discriminants of various extensions of $\mathbb{Q}$ obtained by adjoining roots of unity and radicals.

Throughout this paper, the valuation v_K on any discrete valuation field K is normalized so that the valuation of a uniformizer is 1. The field k is an algebraically closed field of characteristic $p > 0$. We set $K_0 = \text{Frac}(W(k))$ and $K_r = K_0(\zeta_{p^r})$, where ζ_n means a primitive nth root of unity. The absolute ramification index of a finite extension K of K_0 is written e_K.

Our purpose is to study the higher ramification filtrations of certain wild extensions of discrete valuation fields. The main result is the calculation of the higher ramification groups for Galois extensions of the form $K_c(\sqrt[2^c]{a})/K_0$, where $c \geq 1$, $p = 2$, and $a \in K_0$ (Theorem 5.1). In fact, we do not explicitly calculate all of the higher ramification groups, but rather the conductor of the extension, which is the highest index for which there exists a nontrivial higher ramification group for the upper numbering. In principle (and practice), this is enough to calculate all of the higher ramification groups (Proposition 1.3 and the introduction to §5), which is in turn enough to calculate the different and discriminant of the extension ([Ser79, IV, Proposition 4 and VI, §3, Corollary 2]).

Additionally, we calculate an upper bound on the conductor of (the Galois closure of) any extension of the form $K = K_c(\sqrt[p^c]{a})/K_0$, where p is arbitrary and $a \in K_c$, but not necessarily K_0 (Corollary 4.3). In certain situations, we get an exact value for the conductor (Proposition 4.2). Our calculations in this more general situation are in fact used in the proof of Theorem 5.1 (in particular, part (iig)). Our techniques are reminiscent of those used by Viviani in [Viv04], where the assumptions are made that $a \in K_0$ and p is odd. The main idea is to focus on what we call *p-primitive elements* of a mixed characteristic discrete valuation field (Definition 3.3). Extensions obtained by taking roots of such elements are

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particularly amenable to having their higher ramification groups calculated. We then proceed by writing K as the compositum of extensions coming from roots of p-primitive elements and roots of unity, and using theorems about how higher ramification groups behave under taking the compositum (Lemmas 1.1, 1.2).

We have two main motivations. The first comes from [Viv04]. In it, Viviani calculates the higher ramification groups away from 2 of all Galois extensions $\mathbb{Q}(\zeta_m, \sqrt[m]{a})/\mathbb{Q}$, so long as m is odd and $a \in \mathbb{Q}$ satisfies a technical condition. He is able to reduce this to the study of the extensions $\mathbb{Q}_p(\zeta_{p^c}, \sqrt[p^c]{a})/\mathbb{Q}_p$, where p is odd and the p-valuation of $a \in \mathbb{Q}_p$ is either prime to p or divisible by p^c (hereafter, the "valuation condition"). Of course, one can make a base change to the maximal unramified extension $\mathbb{Q}_p^{ur}$ of $\mathbb{Q}_p$ without changing the higher ramification groups. Furthermore, since we are studying algebraic extensions, there is no harm in making a further base change to the completion C of $\mathbb{Q}_p^{ur}$. We note that, if $k = \overline{\mathbb{F}_p}$, then $K_0 = C$. Thus, the calculation of the higher ramification groups in [Viv04] is equivalent to calculating the higher ramification groups of $K_c(\sqrt[p^c]{a})/K_0$ when p is odd, $k = \overline{\mathbb{F}_p}$, and $a \in K_0$ satisfies the valuation condition. Naturally, one would like a similar result when $p = 2$, which is what Theorem 5.1 provides. Furthermore, we need no valuation condition on a when $p = 2$, although we are unfortunately not able to eliminate the valuation condition when p is odd.

The second motivation comes from [Obu09] and [Obu10]. Let $f : Y \to \mathbb{P}^1$ be a G-Galois cover of $\mathbb{P}^1$ branched at 0, 1, and ∞, a priori defined over the algebraic closure of K_0. If a p-Sylow subgroup of G is of order p, then it turns out that f can in fact be defined over a tame extension of K_0 ([Wew03b]). However, if a p-Sylow subgroup of G is cyclic of order p^r, then the best that can be proven at the moment, especially when p is small, is that f can often be defined over a field of the form $K_c(\sqrt[p^c]{a})/K_0$, where $a \in K_c$. In fact, even the stable model of f can often be defined over such an extension. The bounds that we calculate on the conductors of these extensions are sufficient to yield aesthetically pleasing statements of the form "smaller p-Sylow subgroups lead to smaller conductors of the minimal field of definition over K_0") (see [Obu09, Theorem 1.3] and [Obu10, Theorem 1.1] for the specific statements).

After some basic results on how higher ramification groups act under compositums and towers of field extensions (§1 and §2), we study the ramification behavior of prime order Kummer extensions and introduce the concept of p-primitive elements (§3). The technical heart of the paper is §4, where we study the conductor of an extension $K = K_c(\sqrt[p^c]{a})/K_0$ for $a \in K_0$ by breaking this extension up into extensions involving only roots of unity (well understood by [Ser79]) and prime order Kummer extensions. We put everything together in §5 to prove Theorem 5.1.

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1. Higher ramification filtrations

We state here some facts from [Ser79, IV]. Let K be a complete discrete valuation field with residue field k. If L/K is a finite Galois extension of fields with Galois group G, then the group G has a filtration $G = G_0 \geq G_i$ ($i \in \mathbb{R}_{\geq 0}$) for the lower

numbering. If π_K is a uniformizer of K, this filtration is given by

$$g \in G_i \Leftrightarrow v_K(g\pi_K - \pi_K) \geq i + 1.$$

There is also a filtration $G = G^0 \geq G^i$ ($i \in \mathbb{R} \geq 0$) for the upper numbering, defined by $G^i = G_{\psi_{L/K}(i)}$, where $\psi_{L/K} : [0, \infty) \to [0, \infty)$ is a certain monotonically increasing, piecewise linear function ([Ser79, IV, §3]). The inverse of $\psi_{L/K}$ is denoted $\phi_{L/K}$. Clearly $G^{\phi_{L/K}(j)} = G_j$ for $j \in [0, \infty)$. The subgroup G_i (resp. G^i) is known as the *ith higher ramification group for the lower numbering (resp. the upper numbering).* If $H \leq G$, and $M = L^H$, then the ith higher ramification group H_i for the lower numbering for L/M is clearly $G_i \cap H$. If, furthermore, H is normal, then the ith higher ramification group $(G/H)^i$ for the upper numbering for M/K is $G^i/(G^i \cap H) \leq G/H$ ([Ser79, IV, Proposition 14]). We say that the lower numbering is invariant under subgroups, whereas the upper numbering is invariant under quotients.

The *conductor* of L/K, written $h_{L/K}$, is defined by

$$h_{L/K} = \sup_{i \in [0, \infty)} (G^i \neq \{id\}).$$

The *highest lower jump* of L/K, denoted $\ell_{L/K}$, is defined by

$$\ell_{L/K} = \sup_{i \in [0, \infty)} (G_i \neq \{id\}).$$

Of course, $\psi_{L/K}(h_{L/K}) = \ell_{L/K}$ and $\phi_{L/K}(\ell_{L/K}) = h_{L/K}$..

The following lemma is easy (for a proof, see e.g. [Obu09, Lemma 2.3]).

Lemma 1.1. *Given L/K as above, let $M_1, \ldots, M_\ell$ be subfields of L containing K whose compositum is L. Then $h_{L/K} = \max_i(h_{M_i/K})$.*

Lemma 1.2. *Given L/K as above, let M_1, M_2, and M_3 be subfields of L containing K, the compositum of any two of which is L. If $h_{M_1/K} > h_{M_2/K}$, then $h_{M_3/K} = h_{M_1/K} = h_{L/K}$.*

Proof. By Lemma 1.1 applied to M_1 and M_2, we have $h_{L/K} = h_{M_1/K}$. Then the same lemma, applied to M_2 and M_3, implies that $h_{L/K} = h_{M_3/K}$. □

Proposition 1.3. *Given L/K a G-Galois extension as above, the higher ramification filtration is completely determined by knowing the conductor of each Galois extension M/K, where $K \subseteq M \subseteq L$.*

Proof. Clearly it is enough to determine G^i for all $i \geq 0$. For any normal subgroup $H \leq G$, invariance under quotients shows that $H \geq G^i$ iff $(G/H)^i = \{id\}$, that is, if $h_{L^H/K} < i$. There is a unique minimal normal subgroup H such that $(G/H)^i = \{id\}$, as the intersection any two such subgroups is a third such subgroup (this follows from Lemma 1.1). Since G^i is normal in G ([Ser79, IV, Proposition 1]), we conclude that $G^i = H$. □

2. Ramification filtrations in towers

In this section, we give several results about how conductors act in towers.

Lemma 2.1. *Let K be a complete discrete valuation field with algebraically closed residue field. Let $K \subseteq L \subseteq M$ be finite Galois extensions with $G = Gal(M/K)$, $H =$*

Gal(M/L), *and* $G/H =$ *Gal*(L/K). *Then, either* $h_{M/K} = h_{L/K}$ *(in which case* $h_{M/L} \leq \ell_{L/K}$*), or* $h_{M/K} > h_{L/K}$*, in which case*

$$\frac{1}{[L:K]}(h_{M/L} - \ell_{L/K}) = h_{M/K} - h_{L/K}.$$

Proof. Since the upper numbering is invariant under taking quotients, $h_{M/K} \geq h_{L/K}$. Let j be the greatest index such that $G_j \supsetneq H_j$. Now, $G_j = G^{\phi_{M/K}(j)}$. By invariance under quotients, we have that

$$\phi_{M/K}(j) = h_{L/K}. \tag{2.1}$$

By applying $\psi_{L/K}$ to both sides of (2.1), and using the fact that $\phi_{M/K} = \phi_{L/K} \circ \phi_{M/L}$ ([Ser79, IV, Proposition 15]), we obtain that

$$\phi_{M/L}(j) = \ell_{L/K}. \tag{2.2}$$

Suppose that $h_{M/K} = h_{L/K}$. Then, applying $\psi_{L/K}$, we get $\psi_{L/K}(h_{M/K}) = \ell_{L/K}$. But

$$\psi_{L/K}(h_{M/K}) = (\psi_{L/K} \circ \phi_{M/K})(\ell_{M/K}) = \phi_{M/L}(\ell_{M/K}) \geq \phi_{M/L}(\ell_{M/L}) = h_{M/L},$$

so $h_{M/L} \leq \ell_{L/K}$.

Now suppose that $h_{M/K} > h_{L/K}$. Applying $\psi_{M/K}$ to both sides yields

$$\ell_{M/K} > \psi_{M/K}(h_{L/K}) = j,$$

the last equality following from (2.1). The definition of j implies that $G_i = H_i$ for $i > j$, which means that

$$\ell_{M/L} = \ell_{M/K}. \tag{2.3}$$

It also follows that, for all $i > j$, we have $[G : G_i] = [L : K][H : H_i]$, thus the slope of $\phi_{M/L}$ is $[L : K]$ times the slope of $\phi_{M/K}$ for arguments greater than j. So

$$\frac{1}{[L:K]}(\phi_{M/L}(\ell_{M/K}) - \phi_{M/L}(j)) = (\phi_{M/K}(\ell_{M/K}) - \phi_{M/K}(j)). \tag{2.4}$$

By (2.3), $\phi_{M/L}(\ell_{M/K}) = \phi_{M/L}(\ell_{M/L}) = h_{M/L}$, and $\phi_{M/K}(\ell_{M/K}) = h_{M/K}$. Now substituting (2.1) and (2.2) into (2.4) gives the statement of the lemma. □

Corollary 2.2. *Let* K *be a complete discrete valuation field with algebraically closed residue field, and let* $K \subset L \subseteq M$ *be Galois field extensions so that* M *is Galois over* K*. Assume that* $[L : K] = p$*. Then*

$$h_{M/K} = \max(h_{L/K}, \frac{p-1}{p}h_{L/K} + \frac{1}{p}h_{M/L}).$$

Proof. Since $h_{L/K}$ is equal to the (only) lower jump $\ell_{L/K}$ of L/K, Lemma 2.1 implies that either $h_{M/K} = h_{L/K}$ or $\frac{1}{p}(h_{M/L} - h_{L/K}) = h_{M/K} - h_{L/K}$. Solving for $h_{M/K}$ proves the corollary. □

The next corollary generalizes [Ray99, Lemme 1.1.4].

Corollary 2.3. *Let* K *be a complete discrete valuation field with algebraically closed residue field, let* K' *be a* $\mathbb{Z}/p$*-extension of* K*, and let* L *be any finite Galois extension of* K*. Write* L' *for the compositum of* L *and* K' *in some algebraic closure of* K *(we do not assume the extensions are linearly disjoint). Then, if* $h_{L/K} > h_{K'/K}$*, we have* $h_{L'/K'} = ph_{L/K} - (p-1)h_{K'/K}$*. If* $h_{L/K} \leq h_{K'/K}$*, then* $h_{L'/K'} \leq h_{K'/K}$*.*

Proof. We draw a diagram of the situation as follows:

$$\begin{tikzcd}
L \arrow[r, hook, "h_{L'/L}"] & L' \\
K \arrow[u, hook, "h_{L/K}"] \arrow[r, hook, "h_{K'/K}", "\mathbb{Z}/p"'] & K' \arrow[u, hook, "h_{L'/K'}"']
\end{tikzcd}$$

If $h_{L/K} > h_{K'/K}$, then by Lemma 1.1, $h_{L'/K} = h_{L/K} > h_{K'/K}$. In this situation the corollary follows from Lemma 2.1 applied to the tower $K \subseteq K' \subseteq L'$, using the fact that $\ell_{K'/K} = h_{K'/K}$. If $h_{L/K} \leq h_{K'/K}$, then, by Lemma 1.1, we have $h_{L'/K} = h_{K'/K}$. Then $h_{L'/K'} \leq \ell_{K'/K} = h_{K'/K}$ by Lemma 2.1 applied to the tower $K \subseteq K' \subseteq L'$. $\square$

Recall that K_0 is $\mathrm{Frac}(W(k))$, where k is an algebraically closed field of characteristic p, and that $K_r = K_0(\zeta_{p^r})$.

Corollary 2.4. *Let $r \geq 1$, and let L/K_r be a finite extension such that L/K_0 is Galois. Then*

$$h_{L/K_0} = \max\left(r-1, r - \frac{p}{p-1} + \frac{1}{(p-1)p^{r-1}}\left(h_{L/K_r}+1\right)\right).$$

Proof. By [Ser79, Corollary to IV, Proposition 18], the conductor of K_r/K_0 is $r-1$, and the greatest lower jump is $p^{r-1}-1$. So by Lemma 2.1, applied to the tower $K_0 \subset K_r \subset L$, either $h_{L/K_0} = r-1$ or $\frac{1}{(p-1)p^{r-1}}(h_{L/K_r} - (p^{r-1}-1)) = h_{L/K_0} - (r-1)$. Solving for h_{L/K_0} yields the corollary. $\square$

3. Prime order extensions in mixed characteristic

Recall that, if K is a mixed characteristic $(0,p)$ discrete valuation ring, then $e_K = v_K(p)$, the absolute ramification index of K.

Lemma 3.1. *Let K be a finite extension of K_0, and suppose $a = 1 + t \in K$ with $v_K(t) < \frac{p}{p-1}e_K$. Then any pth root of a can be expressed (in an appropriate finite extension L/K) as $1+r$, where $v_L(r) = \frac{v_L(t)}{p}$.*

Proof. Suppose $r \in L$ is such that $1+t = (1+r)^p = 1 + \sum_{i=1}^{p}\binom{p}{i}r^i$. Then $v_L(t) \geq \min_{1\leq i\leq p}(v_L\left(\binom{p}{i}r^i\right))$. Since $v_L(t) < \frac{p}{p-1}e_L$, we have $v_L(r) < \frac{1}{p-1}e_L$, which in turn implies that the minimum is realized for $i = p$. So $v_L(t) = v_L(r^p)$, from which the lemma follows. $\square$

Lemma 3.2. *Let K be a finite extension of K_1, and let L/K be a (nontrivial) $\mathbb{Z}/p$-extension.*

(i) There exists $a \in K$ such that
  - *(a) $L \cong K(\sqrt[p]{a})$*
  - *(b) $a = 1+t$, $0 \leq v_K(t) < \frac{p}{p-1}e_K$, and either $p \nmid v_K(a)$ or $p \nmid v_K(t) > 0$.*

(ii) For a and t as in (i), the conductor $h_{L/K}$ is $\frac{p}{p-1}e_K - v_K(t)$.

Proof. To (i): By Kummer theory, we know we can find a such that $L \cong K(\sqrt[p]{a})$. In choosing a, we are free to multiply by elements of $(K^\times)^p$. Thus we can assume that $0 \leq v_K(a) < p$. If $v_K(a) > 0$, then a clearly satisfies condition (ii) of the lemma. If $v_K(a) = 0$, we can use the fact that k is algebraically closed to choose

$a = 1+t$, where $v_K(t) > 0$. If $v_K(t) \geq \frac{p}{p-1}e_K$, then a is a pth power in K ([Epp73, §0.3]), contradicting the nontriviality of L/K. So $v_K(t) < \frac{p}{p-1}e_K$. If $p \nmid v_K(t)$, then a satisfies condition (ii) of the lemma. If $p | v_K(t)$, then write $t = (\pi_K)^{p\nu}w$, where π_K is a uniformizer of K and $v_K(w) = 0$. Let $y \in K$ be such that $v_K(y^p + w) > 0$ (we can find such a y because k is algebraically closed). If $a' = a(1 + (\pi_K)^\nu y)^p$, then $a' = 1 + t'$ with $v_K(t') > v_K(t)$. So replace a by a' and t by t' and repeat until a satisfies condition (ii) of the lemma. This process must terminate eventually, as the valuation of t is bounded by $\frac{p}{p-1}e_K$.

To (ii): (cf. [Viv04, Theorems 5.6 and 6.3]) We first calculate a uniformizer π_L of L. Then, if σ is a generator of $\text{Gal}(L/K)$, we will determine $v(\sigma(\pi_L - \pi_L))$. Let $\sqrt[p]{a}$ a choice of pth root such that $\sigma(\sqrt[p]{a}) = \zeta_p \sqrt[p]{a}$.

Suppose $p \nmid v_K(a)$. Choose integers m and n such that $mp + nv_K(a) = 1$. Thus $v_L(\pi_K^m a^{n/p}) = 1$, so we set $\pi_L = \pi_K^m a^{n/p}$. Since $v_L(\zeta_p - 1) = \frac{1}{p-1}e_L$, we have $v_L(\sigma(\pi_L) - \pi_L) = mp + v_L(a^{n/p}) + \frac{1}{p-1}e_L$. Since L/K is a $\mathbb{Z}/p$-extension, the definition of the conductor gives $h_{L/K} = mp + \frac{n}{p}v_L(a) + \frac{1}{p-1}e_L - 1 = \frac{p}{p-1}e_K$. Since $v_K(t) = 0$, this proves the proposition in this case.

Now, suppose $p \nmid v_K(t) > 0$. By Lemma 3.1, $v_L(\sqrt[p]{a} - 1) = \frac{v_L(t)}{p} = v_K(t)$. Since $p \nmid v_K(t)$, there exist $m, n \in \mathbb{Z}$ such that $mp + nv_K(t) = 1$. We can even require $0 < n < p$. Then $\pi_L := \pi_K^m(\sqrt[p]{a} - 1)^n$ is a uniformizer of L.

Computing, we find

$$\begin{aligned} \sigma(\pi_L) - \pi_L &= \pi_K^m \left((\sqrt[p]{a} - 1 + (\zeta_p - 1)\sqrt[p]{a})^n - (\sqrt[p]{a} - 1)^n \right) \\ &= \pi_K^m \left(\sum_{i=0}^{n-1} \binom{n}{i} (\sqrt[p]{a} - 1)^i ((\zeta_p - 1)\sqrt[p]{a})^{n-i} \right) \end{aligned}$$

By assumption, $v_L(\zeta_p - 1) = \frac{p}{p-1}e_K > v_K(t) = \frac{v_L(t)}{p} = v_L(\sqrt[p]{a} - 1)$. Also, since $n < p$, it follows that $\binom{n}{i}$ has valuation 0. Thus all terms in the sum above have different valuations, and the term of lowest valuation corresponds to $i = n - 1$. Applying v_L to this term gives $mp + (n-1)v_K(t) + \frac{1}{p-1}e_L = 1 + \frac{p}{p-1}e_K - v_K(t)$. Since L/K is a $\mathbb{Z}/p$-extension, the definition of the conductor gives $h_{L/K} = \frac{p}{p-1}e_K - v_K(t)$. □

Definition 3.3. Let K/K_1 be finite. If $a = 1 + t$ is an element of K such that either $p \nmid v_K(a)$, or $0 < v_K(t) < \frac{p}{p-1}e_K$ and $p \nmid v_K(t)$, then we will say that a is *p-primitive for K*.

Remark 3.4. The proof of Lemma 3.2 shows that if $L \cong K(\sqrt[p]{a})$ is a $\mathbb{Z}/p$-extension of K with $a = 1 + t$, then $h_{L/K} \leq \frac{p}{p-1}e_K - v_K(t)$, with equality holding iff a is p-primitive for K. In particular, if $a' = 1 + t'$ is p-primitive for K and a/a' is a pth power in K, then $v_K(t') \geq v_K(t)$.

Corollary 3.5. *Let L_0 be a finite extension of K_1. Let $a = 1 + t$ be p-primitive for L_0. For $j > 0$, write $L_j = L_0(\sqrt[p^j]{a})$. Then $[L_r : L_0] = p^r$ for $r > 0$, and for $0 < j \leq r$, the conductor of L_j/L_{j-1} is $\frac{p^j}{p-1}e_{L_0} - v_{L_0}(t)$.*

Proof. If $p \nmid v(a)$, then clearly $[L_r : L_0] = p^r$ and $v_{L_0}(t) = 0$. Also, $p \nmid v_{L_{j-1}}(\sqrt[p^{j-1}]{a})$. Then Lemma 3.2 shows that $h_{L_j/L_{j-1}} = \frac{p}{p-1}e_{L_{j-1}} = \frac{p^j}{p-1}e_{L_0}$.

Suppose $v(a) = 0$ and $p \nmid v(t) > 0$. By Lemma 3.2, $[L_1 : L_0] = p$ and h_{L_1/L_0} is as desired. Choose a pth root $\sqrt[p]{a}$. By Lemma 3.1, we have that $v_{L_1}(\sqrt[p]{a} - 1) =$

$\frac{v_{L_1}(t)}{p} = v_{L_0}(t)$, which is prime to p. So $\sqrt[p]{a}$ is p-primitive for L_1. Applying Lemma 3.2 to L_2/L_1 and $\sqrt[p]{a}$ shows that $[L_2 : L_1] = p$ and

$$h_{L_2/L_1} = \frac{p}{p-1}e_{L_1} - v_{L_1}(\sqrt[p]{a} - 1) = \frac{p^2}{p-1}e_{L_0} - v_{L_0}(t),$$

as desired. Repeating this process up to reaching L_r yields the corollary. $\square$

4. Conductors of a certain class of metabelian extensions

Recall that $K_r = K_0(\zeta_{p^r})$. Write v_r for the normalized valuation on K_r such that a uniformizer has valuation 1.

Lemma 4.1. *Choose integers ℓ and c such that $1 \leq \ell \leq c$. Let $a = 1 + t \in K_\ell$ such that a is p-primitive for K_ℓ. Then the conductor of $K_c(\sqrt[p^c]{a})$ over $K_\ell(\sqrt[p^c]{a})$ is less than or equal to $p^{c+\ell-1} - v_\ell(t)$, which is the conductor of $K_\ell(\sqrt[p^c]{a})$ over $K_\ell(\sqrt[p^{c-1}]{a})$.*

Proof. If $c = \ell$, the lemma follows from Corollary 3.5, so assume $c > \ell$. For each i, $0 \leq i \leq c$, let h_i be the conductor of $K_c(\sqrt[p^i]{a})$ over $K_\ell(\sqrt[p^i]{a})$. Then, using [Ser79, IV, Proposition 18], one calculates $h_0 = p^{\ell-1}((c-\ell)(p-1)+1) - 1$. Furthermore, let a_i be the conductor of $K_\ell(\sqrt[p^i]{a})$ over $K_\ell(\sqrt[p^{i-1}]{a})$. By Corollary 3.5, we have $a_i = p^{i+\ell-1} - v_\ell(t)$. Note that $v_\ell(t) < \frac{p}{p-1}e_{K_\ell} = p^\ell$. We must show that $h_c \leq a_c$. Our diagram of field extensions and conductors looks like this:

$$\begin{tikzcd}
K_c \arrow[r, hook] & K_c(\sqrt[p]{a}) \arrow[r, hook] & \cdots \arrow[r, hook] & K_c(\sqrt[p^{c-1}]{a}) \arrow[r, hook] & K_c(\sqrt[p^c]{a}) \\
K_\ell \arrow[u, hook, "h_0"] \arrow[r, hook, "a_1"] & K_\ell(\sqrt[p]{a}) \arrow[u, hook, "h_1"] \arrow[r, hook, "a_2"] & \cdots \arrow[r, hook, "a_{c-1}"] & K_\ell(\sqrt[p^{c-1}]{a}) \arrow[u, hook, "h_{c-1}"] \arrow[r, hook, "a_c"] & K_\ell(\sqrt[p^c]{a}) \arrow[u, hook, "h_c"]
\end{tikzcd}$$

If there exists an i, $0 \leq i < c$, such that $h_i \leq a_{i+1}$, then repeated application of Corollary 2.3 shows that $h_c \leq a_c$. So assume otherwise. Then we have the chain of (in)equalities below (the first comes from repeated application of Corollary 2.3):

$$\begin{aligned}
h_c &= p^c h_0 - p^{c-1}(p-1)a_1 - p^{c-2}(p-1)a_2 - \cdots - p(p-1)a_{c-1} - (p-1)a_c \\
&= p^c h_0 - c(p-1)(p^{c+\ell-1}) + (p^c - 1)v_\ell(t) \\
&= p^{c+\ell-1}(1 - \ell(p-1)) + (p^c-1)v_\ell(t) - p^c \\
&= a_c - \ell(p-1)(p^{c+\ell-1}) + p^c v_\ell(t) - p^c \\
&< a_c - \ell(p-1)(p^{c+\ell-1}) + p^{c+\ell} - p^c \\
&= a_c + p^{c+\ell}(1 - \frac{\ell(p-1)}{p} - \frac{1}{p^\ell}) \\
&\leq a_c.
\end{aligned}$$

$\square$

Proposition 4.2. *Choose integers $\ell \geq 1$ and $c \geq 1$. Let $a = 1 + t \in K_\ell$ such that a is p-primitive for K_ℓ. Write $K = K_{\max(\ell,c)}(\sqrt[p^c]{a})$. Then K is Galois over K_ℓ. Write L for the Galois closure of K over K_0.*

(i) The conductor of K/K_ℓ is $(c(p-1)+1)p^{\ell-1} - v_\ell(t)$.
(ii) The conductor of L/K_0 is $\max(\ell - 1, c + \ell - 1 - \frac{v_\ell(t)-1}{p^{\ell-1}(p-1)})$.

Proof. The extension K/K_ℓ is clearly Galois. Let the a_i be defined as in the proof of Lemma 4.1. Then $a_i = p^{i+\ell+1} - v_\ell(t)$.

To (i): If $c \geq \ell$, then by Corollary 2.2 applied to $K_\ell(\sqrt[p^{c-1}]{a}) \subseteq K_\ell(\sqrt[p^c]{a}) \subseteq K$, and using Lemma 4.1, we have that the conductor of K over $K_\ell(\sqrt[p^{c-1}]{a})$ is a_c. If $c < \ell$, then the conductor of K over $K_\ell(\sqrt[p^{c-1}]{a})$ is a_c by definition. In both cases, applying Corollary 2.2 repeatedly to the extensions $K_\ell(\sqrt[p^{i-1}]{a}) \subseteq K_\ell(\sqrt[p^i]{a}) \subseteq K$ as i ranges from $c-1$ to 1, we obtain that the conductor of K/K_ℓ is

$$\frac{1}{p^{c-1}}a_c + \sum_{i=1}^{c-1} \frac{p-1}{p^i} a_i.$$

This is equal to $(c(p-1)+1)p^{\ell-1} - v_\ell(t)$.

To (ii) Since K is Galois over K_ℓ, its Galois closure L over K_0 is a compositum of conjugate extensions, each with the same conductor over K_ℓ. By Lemma 1.1 and part (i) of this proposition, $h_{L/K_\ell} = h_{K/K_\ell} = (c(p-1)+1)p^{\ell-1} - v_\ell(t)$. By Corollary 2.4, we obtain

$$h_{L/K_0} = \max\left(\ell - 1, c + \ell - 1 - \frac{v_\ell(t) - 1}{p^{\ell-1}(p-1)}\right).$$

□

Corollary 4.3. *Choose integers $\ell \geq 1$ and $c \geq 1$. Let $\alpha \in K_\ell$, not necessarily p-primitive for K_ℓ, with $v_\ell(\alpha) \geq 0$. Write L for the Galois closure of $K := K_{\max(\ell,c)}(\sqrt[p^c]{\alpha})$ over K_0. We know $\alpha = \alpha'\beta^p$, with either $\alpha' = 1$ or α' p-primitive for K_ℓ, and $\beta \in K_\ell$ with $v_\ell(\beta) \geq 0$. Write $\alpha' = 1 + t_{\alpha'}$ and $\beta = 1 + t_\beta$. Then*

$$h_{L/K_0} \leq \mu := \max\left(\max(\ell, c) - 1, c + \ell - 1 - \frac{v_\ell(t_{\alpha'}) - 1}{p^{\ell-1}(p-1)}, c + \ell - 2 - \frac{v_\ell(t_\beta) - 1}{p^{\ell-1}(p-1)}\right).$$

Proof. It is clear that L can be embedded into the Galois closure of $K_{\max(\ell,c)}(\sqrt[p^c]{\alpha'}, \sqrt[p^{c-1}]{\beta})$ over K_0. By Lemma 1.1, it suffices to show that the Galois closures L' of $K_{\max(\ell,c)}(\sqrt[p^c]{\alpha'})$ and L'' of $K_{\max(\ell,c-1)}(\sqrt[p^{c-1}]{\beta})$ over K_0 satisfy $h_{L'/K_0} \leq \mu$ and $h_{L''/K_0} \leq \mu$. By Proposition 4.2 (ii), $h_{L'/K_0} = \max\left(\ell - 1, c + \ell - 1 - \frac{v_\ell(t_{\alpha'})-1}{p^{\ell-1}(p-1)}\right) \leq \mu$ when α' is p-primitive for K_ℓ. By [Ser79, IV, Corollary to Proposition 18], $h_{L'/K_0} = \max(\ell, c) - 1 \leq \mu$ when $\alpha' = 1$.

If $c = 1$ we are done, so assume $c \geq 2$. Pick β', $\gamma \in K_\ell$ such that $\beta = \beta'\gamma^p$ and either $\beta' = 1$ or β' is p-primitive for K_ℓ. Then L'' can be embedded into the compositum of the Galois closures M of $K_{\max(\ell,c-1)}(\sqrt[p^{c-1}]{\beta'})$ and M' of $K_{\max(\ell,c-2)}(\sqrt[p^{c-2}]{\gamma})$ over K_0. If $\beta' = 1$, then $h_{M/K_0} = \max(\ell - 1, c - 2) \leq \mu$. If β' is p-primitive for K_ℓ, then $\beta' = 1 + t_{\beta'}$ with $v_\ell(t_{\beta'}) \geq v_\ell(t_\beta)$. We then have, by Proposition 4.2, that $h_{M/K_0} = \max(\ell - 1, c + \ell - 2 - \frac{v_\ell(t_{\beta'})-1}{p^{\ell-1}(p-1)}) \leq \mu$.

If $c = 2$ we are done, so assume $c \geq 3$. By Lemma 1.1, it remains to prove that $h_{M'/K_0} \leq \mu$. We prove by induction on c that

$$h_{M'/K_0} \leq \max(\ell - 1, c + \ell - 3 + \frac{1}{p^{\ell-1}(p-1)}),$$

which is less than μ because $v_\ell(t_{\alpha'}) < p^\ell$. Write $\gamma = \gamma'\delta^p$, where either $\gamma' = 1$ or γ' is p-primitive for K_ℓ. If $c = 3$, then M' is the Galois closure of $K_{\max(\ell,c-2)}(\sqrt[p^{c-2}]{\gamma'})$ over K_0, and we conclude by Proposition 4.2. If $c > 3$, then M' is contained in the compositum of the Galois closures N of $K_{\max(\ell,c-2)}(\sqrt[p^{c-2}]{\gamma'})$ and N' of $K_{\max(\ell,c-3)}(\sqrt[p^{c-3}]{\delta})$ over K_0. By Proposition 4.2, $h_{N/K_0} \leq \max(\ell - 1, c + \ell - 3 + \frac{1}{p^{\ell-1}(p-1)})$, and by the induction hypothesis, the same holds for h_{N'/K_0}. We conclude using Lemma 1.1. □

The following version of Corollary 4.3 will be useful in §5 and [Obu09].

Corollary 4.4. *Choose integers $\ell \geq 1$ and $c \geq 1$. Let $\alpha \in K_\ell$, not necessarily p-primitive for K_ℓ, with $v_\ell(\alpha) \geq 0$. Write L for the Galois closure of $K := K_d(\sqrt[p^c]{\alpha})$ over K_0, where $d \geq \max(\ell, c)$. Write $\alpha = 1 + t_\alpha$. Then*

$$h_{L/K_0} \leq \mu := \max\left(d - 1, c + \ell - 1 - \frac{v_\ell(t_\alpha) - 1}{p^{\ell-1}(p-1)}, c + \ell - 2 + \frac{1}{p^{\ell-1}(p-1)}\right).$$

Proof. By Remark 3.4, we have $v_\ell(t_{\alpha'}) \geq v_\ell(t_\alpha)$, where $t_{\alpha'}$ is from Corollary 4.3. Furthermore, $v_\ell(t_\beta) \geq 0$, where t_β is from Corollary 4.3. So our corollary follows form Corollary 4.3, along with Lemma 1.1 and the fact that $h_{K_d/K_0} = d-1$ ([Ser79, IV, Corollary to Proposition 18]). □

5. Extensions of the form $K_c(\sqrt[2^c]{a})$

In this section, we assume $p = 2$. In order to understand the higher ramification groups above 2 in an extension $\mathbb{Q}(\zeta_{2^c}, \sqrt[2^c]{a})/\mathbb{Q}$, when $a \in \mathbb{Q}$, it suffices, as mentioned in the introduction, to make a base change to the completion of the maximal unramified extension of $\mathbb{Q}_2$ (this is K_0, when $k = \overline{\mathbb{F}_2}$). We work in the more general context of an extension $K = K_c(\sqrt[2^c]{a})/K_0$, with $a \in K_0$. Note that, since $p = 2$, we have $K_0 = K_1$, so the results of §4 apply.

By Proposition 1.3, in order to determine the higher ramification filtration of K/K_1, we need only determine the conductor of each Galois subextension L of K/K_1. Each such subextension can be written in the form $L = K_{c'}(\sqrt[2^{c''}]{a'})$, where $c' \geq c''$ and $a' \in K_1$. Then Lemma 1.1 implies that $h_{L/K_1} = \max(c' - 1, h_{L'/K_1})$, where $L' = K_{c''}(\sqrt[2^{c''}]{a'})$. Since L'/K_1 is in the same form as our original extension K/K_1, we content ourselves with finding the conductor h_{K/K_1}. For more details on the structure of subextensions of K/K_1, see e.g. [JV90] and [dOV82].

After multiplying a by an element of $(K_1^\times)^{2^c}$, which does not change the extension, we may assume that $0 \leq v_{K_1}(a) < 2^c$ and a is congruent to either 0 or 1 modulo 2. Write $a = 2^n b$, where $v_{K_1}(b) = 0$ and $0 \leq n < 2^c$. After multiplying again by an element of $(K_1^\times)^{2^c}$, we may assume that $b \equiv 1 \pmod 2$.

Theorem 5.1. *Let $K = K_c(\sqrt[2^c]{a})$, where $a = 2^n b \in K_0 = K_1$. Assume that $0 \leq n < 2^c$ and $b \equiv 1 \pmod 2$.*

- *(i) If n is odd, then $h_{K/K_1} = c + 1$.*
- *(ii) Suppose $2|n$.*
  - *(a) If $c = 1$ and $b \equiv 1 \pmod 4$, then $K = K_1$.*
  - *(b) If $c = 1$ and $b \equiv 3 \pmod 4$, then $h_{K/K_1} = 1$.*
  - *(c) If $4|n$, $c > 1$, and $b \equiv 1 \pmod 4$, then $h_{K/K_1} = c - 1$.*
  - *(d) If $4|n$, $c > 1$, and $b \equiv 3 \pmod 4$, then $h_{K/K_1} = c$.*
  - *(e) If $4 \nmid n$, $c > 1$, and $b \equiv 1 \pmod 4$, then $h_{K/K_1} = c$.*

(f) *If* $4 \nmid n$, $c = 2$, *and* $b \equiv 3 \pmod 4$, *then* $h_{K/K_1} = 1$
(g) *If* $4 \nmid n$, $c > 2$, *and* $b \equiv 3 \pmod 4$, *then* $h_{K/K_1} = c - \frac{1}{2}$.

Proof. *To (i):* In this case, a is 2-primitive for K_1, so Proposition 4.2 shows that $h_{K/K_1} = c + 1$.

To (iia): By [Epp73, §0.3], $2^n b$ is a square in K_1.

To (iib): This follows from Lemma 3.2(ii).

To (iic): In this case, K is contained in the compositum of $L := K_c(\sqrt[2^c]{b})$ and $L' := K_{c-2}(\sqrt[2^{c-2}]{2^{n/4}})$. Now, since $b \equiv 1 \pmod 4$, it follows from [Epp73, §0.3] that b is a square in K_1. So $L = K_c(\sqrt[2^{c-1}]{b'})$, where $b'^2 = b$ and $b' \in K_1$. Since $b' \equiv 1 \pmod 2$, it follows from Corollary 4.4 that $h_{L/K} \leq \max(c-1, c-1, c-1) = c-1$.

Also, by Corollary 4.3, $h_{L'/K_1} \leq \max(\max(0, c-3), c-1, c-2) = c-1$. So $h_{K/K_1} \leq h_{LL'/K_1} \leq c-1$, using Lemma 1.1. But K contains K_c and $h_{K_c/K_1} = c-1$ ([Ser79, IV, Corollary to Proposition 18]). So $h_{K/K_1} = c-1$.

To (iid): Consider L and L' as in (iic). Since b is 2-primitive for K_1, Proposition 4.2 gives us that $h_{L/K_1} = c$. We have seen in (iic) that $h_{L'/K_1} \leq c-1$. We conclude using Lemma 1.2, applied to the subextensions L, L', and K of LL'.

To (iie): In this case, K is contained in the compositum of $L := K_c(\sqrt[2^c]{b})$ and $L' := K_{c-1}(\sqrt[2^{c-1}]{2^{n/2}})$. As in (iic), $h_{L/K_1} = c-1$. Also, $2^{n/2}$ is 2-primitive for K_1, so Proposition 4.2(ii) shows that $h_{L'/K_1} = c$. We conclude by applying Lemma 1.2 to the subextensions L', L, and K of LL'.

To (iif): Since -4 is a 4th power in K_2, we have that $K \cong K_2(\sqrt[4]{-b})$. Since $-b \cong 1 \pmod 4$, the result follows from (iic).

To (iig): Since $-4 = (1+i)^4$ (where $i^2 = -1$), it follows that K is contained in the compositum of $L := K_c(\sqrt[2^c]{-2^{n-2}b})$ and the Galois closure L' of $K_{\max(2,c-2)}(\sqrt[2^{c-2}]{1+i})$ over K_1. By (iic), $h_{L/K_1} = c-1$. Also, since $1+i$ is 2-primitive for K_2, Proposition 4.2(ii) shows that $h_{L'/K_1} = c - \frac{1}{2}$. We conclude using Lemma 1.2, applied to the subextensions L', L, and K of LL'. $\square$

References

[dOV82] de Orozco, Mariá Acosta; Vélez, William Y. "The lattice of subfields of a radical extension," J. Number Theory, 15 (1982), no. 3, 388–405.

[Epp73] Epp, Helmut P. "Eliminating wild ramification," Invent. Math., 19 (1973), no. 3, 235–249.

[JV90] Jacobson, Eliot T.; Vélez, William Y. "The Galois group of a radical extension of the rationals," Manuscripta Math., 67 (1990), no. 3, 271–284.

[Obu09] Obus, Andrew. "Fields of moduli of three-point G-covers with cyclic p-Sylow, I," preprint. Available at http://arxiv.org/abs/0911.1103v5.

[Obu10] Obus, Andrew. "Fields of moduli of three-point G-covers with cyclic p-Sylow, II," preprint. Available at http://arxiv.org/abs/1001.3723v5.

[Ray99] Raynaud, Michel. "Specialization des Revêtements en Caractéristique $p > 0$," Ann. Sci. École Norm. Sup., 32 (1999), 87–126.

[Ser79] Serre, Jean-Pierre. "Local Fields," Springer-Verlag, New York, 1979.

[Viv04] Viviani, Filippo. "Ramification Groups and Artin Conductors of Radical Extensions of $\mathbb{Q}$," J. Théor. Nombres Bordeaux, 16 (2004), 779-816.

[Wew03b] Wewers, Stefan. "Three Point Covers with Bad Reduction," J. Amer. Math. Soc. 16 (2003), 991–1032.

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